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(0, 2) Target Space Duality, CICYs and Reflexive Sheaves

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Abstract

It is shown that the recently proposed target space duality for $(0, 2)$ models is not limited to models admitting a Landau-Ginzburg description. By studying some generic examples it is established for the broader class of vector bundles over complete intersections in toric varieties. Instead of sharing a common Landau-Ginzburg locus, a pair of dual models agrees in more general non-geometric phases. The mathematical tools for treating reflexive sheaves are provided, as well.

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1. Introduction

The existence of chiral matter in nature leads one to consider string unifications with $N = 1$ supersymmetry in four space time dimensions. For the heterotic string the condition of $N = 1$ supersymmetry implies $(0, 2)$ supersymmetry on the string world sheet [1]. This class of models has been the subject of study during the last decade [2–12]. Besides the non-perturbative duality to F-theory vacua [13], one recent development in the study of $(0, 2)$ models was the appearance of a perturbative duality first introduced in [6], saying that at large distance completely different looking $(0, 2)$ compactifications can share the same Landau-Ginzburg locus. In [14] it was argued that analogous to mirror symmetry this duality is promoted to a target space duality in the sense that the entire perturbative moduli spaces of a dual pair are isomorphic. The main ingredient for this conclusion was an exact computation of the total dimension of both geometric moduli spaces, in particular including bundle deformations. It turned out, that untouched by a possibly enlarged moduli space at the Landau-Ginzburg locus, the numbers of large radius moduli agree.

The future practical usefulness of this duality crucially depends on how far one will be able to nail down the exact relation between the moduli of a dual pair. This is a difficult question, logically similar to finding the exact form of the mirror map for Calabi-Yau threefolds. In this paper we want to address a more moderate question. Namely to which extend $(0, 2)$ target space duality depends on the special feature of an existing Landau-Ginzburg phase or whether it is a much more general structure for, in general, reflexive sheaves over complete intersections in toric varieties.

As will be verified in section 3, the restriction to Landau-Ginzburg models was rather historically motivated and target space duality extends to much more general $(0, 2)$ models as defined by vector bundles over complete intersections in toric varieties in the first place. Furthermore, as was shown in [9], perturbative $(0, 2)$ vacua can remain finite with sufficiently mild singularities in the bundle as described by reflexive or torsion free sheaves. In section 4 we will present a brief survey of those sheaf theoretic methods needed for the computation of the massless spectrum of such models. It will turn out, that the usual cohomology classes like for instance $H^1(M, V^*)$ do not determine the number of chiral multiplets any longer, instead one has to compute what is called the global extension $\mathrm{Ext}^1(M; \mathcal{V}, \mathcal{O})$. Using those techniques from the theory of coherent sheaves, in section 5 one dual pair of reflexive sheaves over certain base manifolds will be discussed in some

detail. Unfortunately, the explicit calculations turn out to be much more involved than in the bundle case, for the kernels and images of various maps involved in the spectral sequences have to be known in detail. For this reason, exact results for the number of bundle moduli are not yet available, but at least the general method of how to calculate them will be described.

2. Review of $(0,2)$ target space duality

It was known for a while that at large radius completely different $(0,2)$ compactifications of the heterotic string can share the same Landau-Ginzburg locus [6]. Roughly speaking, this is possible because in the context of $(0,2)$ linear sigma models the non-geometric phases in the Kähler moduli space contain less information than the geometric phases. For instance, in a non-geometric phase the chiral fields P_l in the linear sigma model superpotential

$$S = \int d^2z d\theta \left[\Gamma^j W_j(X_i) + P_l \Lambda^a F_a^l(X_i)\right] \tag{2.1}$$

carry a non-vanishing vacuum expectation values, implying the geometric complex and bundle deformations to appear on equal footing. Consequently, an exchange of them can leave the superpotential invariant, whereas at large radius ($P_l = 0$) the model has drastically changed. In [14] it was argued that the first guess, the Landau-Ginzburg locus being a multicritical point, is not convincing and that instead the entire moduli spaces of a dual pair seem to be isomorphic.

More specifically, the quintic threefold $\mathbb{P}_4[5]$ and a dual candidate with base and vector bundle being the resolution of

$$V(1,1,1,2;5) \to \mathbb{P}_{1,1,1,1,1,3}[4,4] \tag{2.2}$$

was studied in detail. The latter model contains a phase for small radii, where it is described by the same Landau-Ginzburg model as the quintic. However, for this model the space time superpotential is flat, so that every modulus of the quintic should correspond to a modulus of the dual $(0,2)$ model. A further confirmation of this picture was found by calculating the dimensions of the total geometric moduli spaces for both models. Indeed in turned out that the sums of complex, Kähler and bundle moduli agree, even though every individual sector is not constant.

A perturbative isomorphy of moduli spaces is what is usually called a target space duality. Well known examples of such dualities include the discrete $R \to 1/R$ symmetries of toroidal compactifications and mirror symmetry in the context of Calabi-Yau compactifications. The latter one is supposed to be correct for a much larger set of threefolds than those exhibiting a Landau-Ginzburg phase or being described by a hypersurface in a weighted projective space, respectively. Thus, it is natural to ask whether $(0,2)$ target space duality can be extended to models not admitting a Landau-Ginzburg phase, as well. For deformations of $(2,2)$ models this leads to complete intersection Calabi-Yau's (CICYs) or, more generally, to complete intersections of hypersurfaces in toric varieties.

3. Target space duality for CICY

In this section we discuss several, we hope sufficiently generic, examples of $(0,2)$ dual pairs, which do not have a Landau-Ginzburg description. Some familiarity with linear sigma models [4], toric geometry [15] and homological algebra [16,3] is assumed in the course of this paper.

3.1. A dual to $\mathbb{P}_5[3,3]$

Using the methods from [14,17], we investigate the model given by the complete intersection of two hypersurfaces of weight three in projective space $\mathbb{P}_5$. The topological data for this model are already known [17]

$$\mathbb{P}_5[3,3], \quad h_{21}=73,\ h_{11}=1,\ h^1(\mathrm{M},\mathrm{End}(\mathrm{T_M}))=140, \tag{3.1}$$

so that there are a total number of **214** geometric moduli. The linear sigma model contains two different phases. For $r>0$ there is a Calabi-Yau phase and for $r<0$ one gets a CY/LG hybrid phase. The hybrid phase can be described as a fiber bundle over $\mathbb{P}_1$, where the fiber over $(p_1,p_2)\in\mathbb{P}_1$ is itself a Landau-Ginzburg model with superpotential

$$W=p_1W_1(x_i)+p_2W_2(x_i). \tag{3.2}$$

The $W_{1,2}$ denote homogeneous polynomials of weight three in six coordinates x_i of weight one. For generic points in $\mathbb{P}_1$ the Landau-Ginzburg model flows in the infrared to a $c=6$ conformal field theory with $\chi=24$. One peculiar feature of this Landau-Ginzburg "K3" is its rigidness in the sense that it does not have any Kähler deformation. In general it is not

known how to compute any further data in hybrid phases, but by using a trick, one can gain some pieces of information. The "K3" Landau-Ginzburg model becomes singular over exactly six points in the base, so that it is equivalent to another "K3" fibration written as $\mathbb{P}_{1,1,2,2,2,2,2}[6]$. This latter model also is the rigid "K3" fibered over a $\mathbb{P}_1$ with six singular fibers. However, this model has a Landau-Ginzburg phase and the calculation of the Hodge numbers, $h_{21} = 73$ and $h_{11} = 1$, indeed gives the desired result, The total number of moduli for $\mathbb{P}_{1,1,2,2,2,2,2}[6]$ comes out as 244, but it is not clear that this really captures the number of moduli in the hybrid phase.

The base manifold of a dual $(0,2)$ model can be obtained by performing a (formal) conifold transition on $\mathbb{P}_5[3,3]$. Going to a point in complex structure moduli space, where the hypersurface equations are

$$W_1 = P_3(x_i), \qquad W_2 = x_1 F_2(x_i) - x_2 G_2(x_i), \tag{3.3}$$

and making a small resolution gives

$$\begin{aligned} W_1 = P_3(x_i), \qquad W_2 &= x_1 y_1 + G_2(x_i) y_2 \\ W_3 &= x_2 y_1 + F_2(x_i) y_2. \end{aligned} \tag{3.4}$$

This can be recognized as the intersection of three hypersurfaces in the toric variety given by the C^* actions

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Γ_1	Γ_2	Γ_3
1	0	0	0	0	0	1	0	0	-1	-1
1	1	1	1	1	1	0	1	-3	-2	-2

Table 3.1: *Charges for the base*

Using toric methods[1] one obtains that the intersection ring on the CICY is

$$3\eta_1^3 - 3\eta_1^2\eta_2 + 3\eta_1\eta_2^2 + 9\eta_2^3, \tag{3.5}$$

where $\eta_{1,2}$ denote the two independent sections of the base[2]. The Euler number turns out to be $\chi = -120$. Using the algorithm to compute the cohomology classes of line bundles in the

[1] For some of the calculations involving toric varieties the maple packages *Schubert* and *Puntos* have been used [18]

[2] The negative intersection numbers in (3.5) appear because the charges in Table 3.1 are not the Mori vectors

ambient space [9,14] and tracing through the long exact sequences in bundle cohomology, one obtains for the detailed Hodge numbers $h_{21} = 62$ and $h_{11} = 2$. The resolution of the vector bundle gives

λ_1	λ_2	λ_3	λ_4	λ_5	λ_1	p_1	p_2
1	0	0	0	0	0	0	-1
0	2	1	1	1	1	-3	-3

Table 3.2: *Charges for the bundle*

This bundle is defined with one fermionic gauge symmetry, so that its rank is indeed three and the resulting gauge group E_6. It is given by the cohomology of the monad

$$0 \to \mathcal{O}|_M \to \mathcal{O}(1,0) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(0,1)^4|_M \to \mathcal{O}(0,3) \oplus \mathcal{O}(1,3)|_M \to 0. \tag{3.6}$$

The third Chern class is $c_3(V_M) = -144$, which indeed comes from $h^1(M, V_M) = 73$ generations and $h^2(M, V_M) = 1$ antigenerations. The tedious computation of the bundle deformations is straightforward and leads to exactly $h^1(M, \mathrm{End}(V_M)) = 150$ additional moduli. Thus, the total number of moduli is **214**, which agrees nicely with the $\mathbb{P}_5[3,3]$ results.

Now, the question is, whether one can find a phase for this $(0,2)$ model, which coincides with the $(0,2)$ deformation of the hybrid phase of $\mathbb{P}_5[3,3]$. The model under consideration has five different phases, two of them geometric the other three non-geometric. After a tedious consideration one finds that for $r_1 > 0$ and $r_2 < 0$ the model is described by indeed the same hybrid phase. Apparently, the reason is, that in non-geometric phases the p fields in the linear sigma model carry non-zero vacuum expectation value implying that there does not exist a distinction between complex and bundle moduli in the superpotential. The natural conclusion is that $(0,2)$ target space duality is not a special feature of the rather restricted set of models containing a Landau-Ginzburg phase, but carries over to the much broader class of vector bundles over complete intersections in toric varieties.

As we have seen, starting with a deformation of a $(2,2)$ model, the base manifold of a $(0,2)$ dual can be obtained by performing a conifold transition on the former threefold. However, this should be regarded as a recipe only for getting the base of the dual, we are not claiming that there is indeed a transition. This would not make sense, since the dual pair is supposed to be isomorphic, anyway. It simply means, that starting with a $(2,2)$ model and making a conifold transition, on the resolved base space one can define a bundle such that the former and latter model are isomorphic.

3.2. A dual to a hypersurface in $\mathbb{P}_3 \times \mathbb{P}_1$

Another type of threefolds which do not contain a pure Landau-Ginzburg description, are hypersurfaces in products of projective spaces. In particular, the model

$$\begin{matrix} \mathbb{P}_3 \\ \mathbb{P}_1 \end{matrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \tag{3.7}$$

is studied with Hodge numbers $h_{21} = 86$ and $h_{11} = 2$. The computation of bundle deformations gives $h^1(M, \mathrm{End}(T_M)) = 188$, so that the total number of moduli at large radius is **276**. The model has three phases, one Calabi-Yau and two LG/CY hybrid phases. In one of the hybrid phases one has a geometric $\mathbb{P}_1$ base with homogeneous coordinates $y_{1,2}$. Fibered over this space is the "K3", $\mathbb{P}_3[4]$, in its Landau-Ginzburg phase. The superpotential looks like

$$W = \sum_{i=1}^{4} p_i^{(2)}(y_1, y_2)\ x_i^4, \tag{3.8}$$

where the $p_i^{(2)}$ denote homogeneous polynomials of degree two. Thus, one has a "K3" fibration with generically eight singular fibers. One encounters the same situation for the K3 fibration $\mathbb{P}_{1,1,2,2,2}[8]$, which consistently has Hodge numbers $(h_{21}, h_{11}) = (86, 2)$, as well. The total number of moduli for the latter model is 294, but as above it is not clear whether this really counts the number of moduli seen in the hybrid phase. The second hybrid phase is a discrete $\mathbb{Z}_2$ bundle over $\mathbb{P}_3$. On the boundary of these two hybrid phases one finds a gauged Landau-Ginzburg phase.

The data of the $(0, 2)$ dual model can be obtained straightforwardly. The base is the threefold defined by the C^* actions in Table 3.3

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Γ_1	Γ_2
2	1	1	1	1	0	0	0	-3	-3
2	0	0	0	0	1	1	0	-2	-2
1	0	0	0	0	0	0	1	-1	-1

Table 3.3: *Charges for the base*

Computing the intersection ring for the toric variety gives

$$2\eta_1^3 + 4\eta_1^2\eta_2 + \eta_1\eta_2\eta_3 - 2\eta_1\eta_3^2 - 2\eta_2\eta_3^2 + 8\eta_3^3, \tag{3.9}$$

which allows one to calculate the Euler number $\chi = -144$. The more refined cohomology calculation reveals $(h_{21}, h_{11}) = (75, 3)$. The bundle on the threefold is described by the data in Table 3.4

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	p
0	1	1	0	0	2	-4
0	0	0	1	1	0	-2
1	0	0	0	0	0	-1

Table 3.4: *Charges for the bundle*

The bundle is defined with two fermionic gauge symmetries as the cohomology of the monad

$$0 \to \mathcal{O}^2|_M \to \mathcal{O}(0,0,1) \oplus \mathcal{O}(1,0,0)^2 \oplus \mathcal{O}(0,1,0)^2 \oplus \mathcal{O}(2,0,0)|_M \to \mathcal{O}(4,2,1)|_M \to 0. \tag{3.10}$$

The bundle valued cohomology computation gives $h^1(M,V_M) = 86$, $h^2(M,V_M) = 2$ and $h^1(M,\mathrm{End}(V_M)) = 198$, so that the number of moduli indeed adds up to **276**. This $(0,2)$ model has six different phases, three of them non-geometric and the sector spanned by the vertices

$$\{(-4,-2,-1),(0,1,0),(0,0,1)\} \tag{3.11}$$

in the secondary fan, contains a phase which is the same CY/LG hybrid (3.8) as for the $(0,2)$ deformation of the original model (3.7). It can be checked that the second CY/LG hybrid phase of (3.7) corresponds to the sector

$$\{(-4,-2,-1),(1,0,0),(0,0,1)\} \tag{3.12}$$

in the Kähler moduli space and that on the boundary one gets a gauged Landau-Ginzburg phase.

3.3. A strict $(0,2)$ dual pair

So far, solely dual pairs with one model being the deformation of a $(2,2)$ model have been studied. In these cases one new Kähler class is introduced in the base. We further pursue a model which has been introduced in [10] and lives on the base ambient space $\mathbb{P}_3 \times \mathbb{P}_2$. Model A is defined by the data in Table 3.5 for the base

x_1	x_2	x_3	x_4	x_5	x_6	x_7	Γ_1	Γ_2
1	1	1	1	0	0	0	-2	-2
0	0	0	0	1	1	1	-2	-1

Table 3.5: *Charges for the base*

and the data in Table 3.6 for the vector bundle

λ_1	λ_2	λ_3	λ_4	p
0	0	1	2	-3
1	1	0	0	-2

Table 3.6: *Charges for the bundle*

This model is consistent without any fermionic gauge symmetry, so that the rank three bundle is simply given by the exact sequence

$$0 \to V_M \to \mathcal{O}(0,1)^2 \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(2,0)|_M \to \mathcal{O}(3,2)|_M \to 0 \tag{3.13}$$

shortening the bundle cohomology calculations considerably. One gets $h^1(M, V_M) = 84$ generations in the **27** representation of E_6 and no antigeneration. Adding up the Hodge numbers of the base, $h_{21} = 62$ and $h_{11} = 2$, and the bundle deformations, $h^1(M, \mathrm{End}(V_M)) = 184$ one obtains the dimension of the total moduli space **248**.

After exchanging $\{W_1, W_2\} \leftrightarrow \{F_1, F_4\}$ the data for the base of the dual model B are

x_1	x_2	x_3	x_4	x_5	x_6	x_7	Γ_1	Γ_2
1	1	1	1	0	0	0	-3	-1
0	0	0	0	1	1	1	-1	-2

Table 3.7: *Charges for the base*

The vector bundle is defined by the charges in Table 3.8

λ_1	λ_2	λ_3	λ_4	p
0	1	1	1	-3
1	0	0	1	-2

Table 3.8: *Charges for the bundle*

The charged matter spectrum turns out to be same as for model A, $h^1(M, V_M) = 84$ and $h^2(M, V_M) = 0$. The neutral matter receives contributions from $h_{21} = 59$ complex, $h_{11} = 2$ Kähler and $h^1(M, \mathrm{End}(V_M)) = 187$ bundle deformations. As expected, the total number of moduli, **248**, is identical to the result for model A. One can further show, that model A and model B are identical in their two CY/LG hybrid phases.

We hope, that the above examples have convinced the reader that $(0, 2)$ target space duality is a general pattern in the class of $(0, 2)$ models and not a rare exception for those models allowing a Landau-Ginzburg description. It should be clear, that something deep in mathematics is going on here. There should exist a duality map acting on vector bundles

over toric varieties so that the sum $h_{21} + h_{11} + h^1(M, \mathrm{End(V)})$ is invariant. This is very similar to mirror symmetry for $(2,2)$ models, where $h_{21}+h_{11}$ is constant. The right way to address these question in mathematical terms is probably to find a combinatoric description of coherent sheaves over toric varieties which also includes the bundle deformations.

4. Reflexive sheaves

In [9] it was nicely shown in the framework of linear sigma models that $(0,2)$ models can live with some mild singularities in the bundle. In particular, the defining maps F_a in a sequence

$$0 \to V_M \to E_M \stackrel{F_a}{\to} \mathcal{O}(D)|_M \to 0 \tag{4.1}$$

are allowed to vanish on a codimension three locus S in the threefold M. In (4.1) E_M denotes a vector bundle and $\mathcal{O}(D)$ the line bundle associated to the divisor $D \subset M$. It was shown that the parameter space of the linear sigma model remains compact by gluing in some proturberances at the singularities. As a consequence, in the large radius limit the sequence (4.1) is no longer exact, but can be extended to an exact sequence by including the cokernel of the map F_a

$$0 \to V_M \to E_M \stackrel{F_a}{\to} \mathcal{O}(D)|_M \to \mathcal{O}(D)|_S \to 0. \tag{4.2}$$

Thus, the coherent sheaf V_M fails to be locally free over a sublocus of codimension three in the threefold M. For the cases studied in this paper, the sublocus S is describable as the complete intersection of three hypersurfaces $S = \{f_1 = f_2 = f_3 = 0\}$. The ideal generated by these f_is is denoted as I. Then $\mathcal{O}(D)|_S$ fits into the exact sequence

$$0 \to I \otimes \mathcal{O}(D)|_M \to \mathcal{O}(D)|_M \to \mathcal{O}(D)|_S \to 0. \tag{4.3}$$

As explained in the following subsection, for sheaves of type (4.2) reflexivity still holds in the sense $V_M^{**} = V_M$. A sheaf like $\mathcal{O}(D)|_S$ supported only at a finite number of points is called a skyscraper sheaf. Many of the salient features of vector bundles do not generalize trivially to coherent sheaves but there exists a nice mathematical theory describing the peculiarities occurring for this latter structure. For mathematical details of the following brief digression on coherent sheaves the reader is referred to the existing literature [16]. A good introduction to sheaf theory for physicist has been presented in [3]. We do not repeat everything mentioned in [3], but continue their introduction and focus on those technical tools needed for the practical purpose of calculating the massless spectrum of $(0,2)$ string models. A less extensive survey on coherent sheaves has also been provided recently in [19].

4.1. Coherent sheaves

A sheaf $\mathcal{F}$ of $\mathcal{O}$ modules over a complex manifold M is said to be *coherent*, if locally it has a presentation

$$\mathcal{O}^{(p)} \to \mathcal{O}^{(q)} \to \mathcal{F} \to 0. \tag{4.4}$$

The crucial property for the following is that a coherent sheaf $\mathcal{F}$ always allows what is called a *global syzygy* or a *locally free resolution*,

$$E.(\mathcal{F}) : 0 \xrightarrow{\delta} E_n \xrightarrow{\delta} \ldots \xrightarrow{\delta} E_1 \xrightarrow{\delta} \mathcal{F} \to 0, \tag{4.5}$$

where all sheaves E_j are locally free (vector bundles) and the sequence is exact. For vector bundles one is used to the fact, that tensoring with another vector bundle or taking the dual of a short exact sequence yields another short exact sequence. This is not any longer true for coherent sheaves. The way in which it fails is measured by the sheaves *(local) extension*, $\underline{Ext}^n(\mathcal{F}, \mathcal{G})$, and *(local) torsion*, $\underline{Tor}_n(\mathcal{F}, \mathcal{G})$, defined as the cohomology and homology

$$\begin{aligned} \underline{Ext}^n(\mathcal{F}, \mathcal{G}) &= H^n_\delta\left(Hom(E.(\mathcal{F}), \mathcal{G})\right) \\ \underline{Tor}_n(\mathcal{F}, \mathcal{G}) &= H^\delta_n\left(E.(\mathcal{F}) \otimes \mathcal{G}\right), \end{aligned} \tag{4.6}$$

respectively. From (4.5) it is obvious that $\underline{Ext}^0(\mathcal{F}, \mathcal{G}) = Hom(\mathcal{F}, \mathcal{G})$ and $\underline{Tor}_0(\mathcal{F}, \mathcal{G}) = \mathcal{F} \otimes \mathcal{G}$. Moreover, as one is used to from ordinary cohomology, short exact sequences of sheaves

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \tag{4.7}$$

imply long exact sequences of $\underline{Ext}^n$ and $\underline{Tor}_n$. For instance, the long exact sequence for local extension looks like

$$0 \to Hom(\mathcal{F}, \mathcal{M}) \to Hom(\mathcal{G}, \mathcal{M}) \to Hom(\mathcal{H}, \mathcal{M}) \to \underline{Ext}^1(\mathcal{F}, \mathcal{M}) \to \underline{Ext}^1(\mathcal{G}, \mathcal{M}) \to \ldots \tag{4.8}$$

showing in which sense (local) $\underline{Ext}$ measures the extend to which $Hom(\,.\,, \mathcal{M})$ fails to be exact. Since for a locally free sheaf one can choose $E_0 = \mathcal{F}$ with all other bundles E_is vanishing, it is clear that the higher extensions $\underline{Ext}^n(\mathcal{F}, \mathcal{G})$ vanish for $n > 0$, so that one recovers the familiar features of vector bundles.

As an application of working with local extensions it is shown that sheaves defined by an exact sequence like (4.2) are indeed reflexive. Joining the sequences (4.2) and (4.3) into the diagram

$$\begin{array}{c} \qquad\qquad\qquad\qquad 0 \\ \qquad\qquad\qquad\qquad \downarrow \\ 0 \to V_M \to E_M \to I \otimes \mathcal{O}(D)|_M \to 0 \\ \qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad \mathcal{O}(D)|_M \\ \qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad \mathcal{O}(D)|_S \\ \qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad 0 \end{array} \tag{4.9}$$

and applying the $\underline{Ext}^q(\,.\,,\mathcal{O})$ functor leads to the diagram

$$\begin{array}{l} \qquad \underline{Ext}^2(\mathcal{O}(D)|_S,\mathcal{O}) \\ \qquad \uparrow \\ \qquad \underline{Ext}^1(I\otimes\mathcal{O}(D)|_M,\mathcal{O}) \\ \qquad \uparrow \\ \qquad \underline{Ext}^1(\mathcal{O}(D)|_M,\mathcal{O}) \\ \qquad \uparrow \\ \qquad \underline{Ext}^1(\mathcal{O}(D)|_S,\mathcal{O}) \\ \qquad \uparrow \\ 0 \to Hom(I\otimes\mathcal{O}(D)|_M,\mathcal{O}) \to E_M^* \to V_M^* \to \underline{Ext}^1(I\otimes\mathcal{O}(D)|_M,\mathcal{O}) \to 0 \\ \qquad \uparrow \\ \qquad Hom(\mathcal{O}(D)|_M,\mathcal{O}) \\ \qquad \uparrow \\ \qquad Hom(\mathcal{O}(D)|_S,\mathcal{O}) \\ \qquad \uparrow \\ \qquad 0 \end{array} \tag{4.10}$$

Using the Koszul resolution, it is proven in [16] that if S has codimension n, one gets for the local extensions

$$\underline{Ext}^q(\mathcal{O}_S,\mathcal{O}) = 0 \qquad \text{for } q \in \{0,\dots,n-1\}. \tag{4.11}$$

Using this for our codimension three locus and that for locally free sheaves $\underline{Ext}^q$ vanishes

for $q > 0$ the diagram (4.10) collapses to the short exact sequence[1]

$$0 \to \mathcal{O}^*(D)|_M \to E_M^* \to V_M^* \to 0. \tag{4.12}$$

Dualizing (4.12) and using reflexivity of locally free sheaves leads us back to the sequence

$$0 \to V_M^{**} \to E_M \to \mathcal{O}(D)|_M \to \mathcal{O}(D)|_S \to 0, \tag{4.13}$$

implying the desired result $V_M^{**} = V_M$.

One essential property of vector bundles over compact, complex manifolds is Serre duality

$$H^p(M, V)^* \cong H^{n-p}(M, V^* \otimes \mathcal{K}_M) \tag{4.14}$$

with $\mathcal{K}_M$ denoting the canonical bundle of M. The bundle valued cohomologies are understood as global Čech cohomologies. In order to formulate the sheaf theoretic generalization one introduces so called global Ext. It is defined as the hypercohomology of the complex of sheaves $Hom(E.(\mathcal{F}), \mathcal{G})$ over M:

$$\mathrm{Ext}(M; \mathcal{F}, \mathcal{G}) = \mathbb{H}^*(M; Hom\,(E.(\mathcal{F}), \mathcal{G})) \tag{4.15}$$

In general, the hypercohomolgy of a complex of sheaves $\mathcal{K} = (\mathcal{K}^*, d)$ over a manifold M is defined as the cohomology of the associated single complex $(C^*(\underline{U}, \mathcal{K}), D = \delta + d)$ of the double complex

$$\{C^p(\underline{U}, \mathcal{K}^q); \delta, d\}\,. \tag{4.16}$$

Here $C^p(\underline{U}, \mathcal{K}^q)$ denotes the Čech cochains of degree p with values in the sheaf $\mathcal{K}^q$ and δ is the Čech coboundary operator. One of the two abuting spectral sequences of the double complex (4.16) leads after two steps to

$$\begin{aligned} E_2^{p,q} &= H^p(M, \underline{Ext}^q(\mathcal{F}, \mathcal{G})) \\ E_\infty^{p,q} &\Rightarrow \mathrm{Ext}^{p+q}(M; \mathcal{F}, \mathcal{G}). \end{aligned} \tag{4.17}$$

Now, it has become evident that for $\mathcal{F}$ a locally free sheaf, one obtains that global Ext reduces to the ordinary cohomology groups

$$\mathrm{Ext}^q(M, \mathcal{F}, \mathcal{G}) \cong H^q(M, \mathcal{F}^* \otimes \mathcal{G}). \tag{4.18}$$

[1] Note, that a map between coherent sheaves can be injective without being injective on each fiber

In particular for the structure sheaf $\mathcal{O}$ (4.18) means $\mathrm{Ext}^q(M, \mathcal{O}, \mathcal{G}) \cong H^q(M, \mathcal{G})$.

What is needed in the following section is a tool of how to calculate global $\mathrm{Ext}(M; \mathcal{F}, \mathcal{G})$ for a sheaf $\mathcal{F}$ fitting into an exact sequence of sheaves

$$0 \to \mathcal{H}_n \xrightarrow{\eta} \mathcal{H}_{n-1} \xrightarrow{\eta} \ldots \xrightarrow{\eta} \mathcal{H}_0 \to \mathcal{F} \to 0 \tag{4.19}$$

where for each individual $\mathcal{H}_i$ global Ext is known. Note, that it is not required that the $\mathcal{H}_i$s are locally free. As usual in homological algebra one considers the double complex $\{C^p(\underline{U}, Hom(\mathcal{H}_q, \mathcal{G})); D, \eta\}$ and running the two abutting spectral sequences allows one to express global $\mathrm{Ext}(M; \mathcal{F}, \mathcal{G})$ in terms of $\mathrm{Ext}(M; \mathcal{H}_q, \mathcal{G})$. One explicit example will be discussed in the following section.

Finally, Serre duality for a general coherent sheaf becomes

$$H^p(M, \mathcal{F})^* \cong \mathrm{Ext}^{n-p}(M; \mathcal{F}, \mathcal{K}_M). \tag{4.20}$$

This closes the compact digression on coherent sheaves, in which the necessary technical tools for dealing with reflexive sheaves of the kind (4.2) have been provided.

4.2. Example of a reflexive sheaf

In this subsection the methods introduced in the last subsection will be applied to the determination of the charged massless spectrum of a specific $(0, 2)$ model. Before that the relation between the different massless modes of the string theory and the various cohomology groups of the sheaf over the Calabi-Yau manifold M has to be clarified. For a vector bundle V the number of generations and antigenerations were determined by $H^1(M, V)$ and $H^1(M, V^*)$. Furthermore, some of the uncharged chiral fields corresponded to the traceless part of $H^1(M, Hom(V, V))$. In view of (4.18) and Serre duality (4.20) the generalization to a coherent sheaf $\mathcal{V}$ appears to be that $\mathrm{Ext}^1(M; \mathcal{O}, \mathcal{V})$ and $\mathrm{Ext}^1(M; \mathcal{V}, \mathcal{O})$ count the generations and antigenerations, respectively. Then it seems natural, that the part of the number of uncharged singlets is related to $\mathrm{Ext}^1(M; \mathcal{V}, \mathcal{V})$.

Consider the following two singular configurations

$$\begin{aligned} A:&\ \ V(1,1,2,4;8) \to \mathbb{P}_{1,1,1,3,3,3}[6,6] \\ B:&\ \ V(1,1,2,2,2;8) \to \mathbb{P}_{1,1,1,3,3,3}[8,4], \end{aligned} \tag{4.21}$$

the resolution of which is supposed to lead to a dual pair of $(0, 2)$ models. Both models share the same Landau Ginzburg locus, for which one obtains $N_{27} = 96$ generations and $N_{\overline{27}} = 4$ antigenerations. The resolution of the base manifold of model A gives

x_1	x_2	x_3	x_4	x_5	x_6	x_7	Γ_1	Γ_2
1 3	1 3	1 3	0 1	0 1	1 0	0 1	-2 -6	-2 -6

Table 4.1: *Charges for the base*

with Euler number $\chi = -144$ resulting from $h_{21} = 77$ complex deformations and $h_{11} = 5$ Kähler deformations. A possible resolution of the bundle is given by the data in Table 4.2

λ_1	λ_2	λ_3	λ_4	p
0 1	1 1	0 2	2 4	-3 -8

Table 4.2: *Charges for the sheaf*

without any fermionic gauge symmetry. Naively, the third Chern class of the bundle comes out as $c_3(V_M) = -192$ which is not what one expects from the Landau-Ginzburg computation. The reason for this mismatch is that V actually is not a bundle but a reflexive sheaf. This can be seen by looking at the functions F_a in more detail. Since they have to be of the form

$$\begin{aligned} F_1 &= x_6\, p_{(2,7)}, \quad F_2 = p_{(2,7)} \\ F_3 &= x_6\, p_{(2,6)}, \quad F_4 = p_{(1,4)}, \end{aligned} \tag{4.22}$$

they simultaneously vanish on the complete intersection of the three divisors $S = \{x_6 = p_{(2,7)} = p_{(1,4)} = 0\} \subset M$. The set S consists of exactly four points. More formally this can be seen by using the Koszul sequence for the structure sheaf on S

$$\begin{aligned} &0 \to \mathcal{O}(-4,-11)|_M \to \mathcal{O}(-3,-11) \oplus \mathcal{O}(-3,-7) \oplus \mathcal{O}(-2,-4)|_M \to \\ &\to \mathcal{O}(-2,-7) \oplus \mathcal{O}(-1,0) \oplus \mathcal{O}(-1,-4)|_M \to \mathcal{O}|_M \to \mathcal{O}_S \to 0 \end{aligned} \tag{4.23}$$

to compute $h^0(M, \mathcal{O}_S) = 4$.

It is worth mentioning, that in the Calabi-Yau phase of the linear sigma $\{r_1 > 0, r_2 > 3r_1\}$ vanishing of the D-terms

$$\begin{aligned} |x_1|^2 + |x_2|^2 + |x_3|^2 + |x_6|^2 - 3|p|^2 &= r_1 \\ |x_4|^2 + |x_5|^2 + |x_7|^2 - 3|x_6|^2 + |p|^2 &= r_2 - 3r_1 \end{aligned} \tag{4.24}$$

still forces the parameter space to be compact. Over the singular set S the field p is no longer set to zero, but partly parameterizes four new $\mathbb{P}_1$ proturberances glued in automatically to resolve the singularity. Thus, as already observed in [9] such mild singularities

in the vector bundle do not lead to singularities in the conformal field theory and string theory is able to resolve them even on the perturbative level.

The reflexive sheaf $\mathcal{V}_M$ is defined via the exact sequence

$$0 \to \mathcal{V}_M \to \mathcal{O}(0,1) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(2,4)|_M \to \mathcal{O}(3,8)|_M \to \mathcal{O}(3,8)|_S \to 0. \quad (4.25)$$

In order to determine the number of generations $\mathrm{Ext}^1(M; \mathcal{O}, \mathcal{V}_M)$ and antigenerations $\mathrm{Ext}^1(M; \mathcal{V}_M, \mathcal{O})$ one has to run the spectral sequence. After determining all the intermediate cohomology classes, one finally arrives at the spectral sequence

	$\mathcal{V}_M$	$\bigoplus \mathcal{O}(m,n)\|_M$	$\mathcal{O}(3,8)\|_M$	$\mathcal{O}(3,8)\|_S$
$\mathrm{Ext}^0(M; \mathcal{O}, .)$	0	36	132 $\xrightarrow{\alpha}$	4
$\mathrm{Ext}^1(M; \mathcal{O}, .)$	96	0	0	0
$\mathrm{Ext}^2(M; \mathcal{O}, .)$	4	0	0	0
$\mathrm{Ext}^3(M; \mathcal{O}, .)$	0	0	0	0

Table 4.3: *Spectral sequence for determining Ext*$^1(M; \mathcal{O}, \mathcal{V}_M)$

Here, it has been used that the image of the map α has dimension zero, a fact which can be seen simply by observing that every section of $\mathcal{O}(3,8)|_M$ must contain at least one x_6 and thus vanishes when restricted to S. As the first column in Table 4.3 shows, the number of charged chiral multiplets agrees with the Landau-Ginzburg result.

As already mentioned, generally for reflexive sheaves one is forced to study some of the maps involved in the sequences in detail. This makes live much harder than in the bundle case, where it often suffices to know merely the dimensions of the cohomology groups without the action of the various maps. Even though the number of antigenerations $\mathrm{Ext}^1(M; \mathcal{V}, \mathcal{O})$ is determined by Serre duality, let us verify it explicitly by using the formalism of coherent sheaves. By using $\mathrm{Ext}^q(M; \mathcal{L}, \mathcal{O}) = H^q(M, \mathcal{L}^*)$ for the line bundles involved and $\mathrm{Ext}^q(M; \mathcal{O}(3,8)|_S, \mathcal{O}) = \mathrm{Ext}^{3-q}(M; \mathcal{O}, \mathcal{O}(3,8)|_S)$ for the skycraper sheaf, one obtains the following spectral sequence

	$\mathcal{O}(3,8)\|_S$	$\mathcal{O}(3,8)\|_M$	$\bigoplus \mathcal{O}(m,n)\|_M$	$\mathcal{V}_M$
$\mathrm{Ext}^0(M; ., \mathcal{O})$	0	0	0	0
$\mathrm{Ext}^1(M; ., \mathcal{O})$	0	0	0	4
$\mathrm{Ext}^2(M; ., \mathcal{O})$	0	0	0	96
$\mathrm{Ext}^3(M; ., \mathcal{O})$	4 $\xrightarrow{\alpha^t}$	132	36	0

Table 4.4: *Spectral sequence for determining Ext*$^1(M; \mathcal{V}_M, \mathcal{O})$

Note again, that this is definitely different from $H^q(M, \mathcal{V}_M^*)$ with $(h^0, \ldots, h^3) = (0, 0, 96, 0)$. Due to the necessity of determining the kernels and images of various maps, the computation of $\mathrm{Ext}^q(M; \mathcal{V}_M, \mathcal{V}_M)$ is fairly involved, even though in principal, the algorithm described in [14] for vector bundles carries over as long as one carefully works with extensions instead of simply with bundle valued cohomologies.

The dual model B can be treated completely analogously. The resolution of the base manifold gives

x_1	x_2	x_3	x_4	x_5	x_6	x_7	Γ_1	Γ_2
1	1	1	0	0	1	0	−3	−1
3	3	3	1	1	0	1	−8	−4

Table 4.5: *Charges for the base*

with Euler number $\chi = -208$ coming from $h_{21} = 109$ complex deformations and $h_{11} = 5$ Kähler deformations. A possible resolution of the bundle is given by the data in Table 4.6

λ_1	λ_2	λ_3	λ_4	λ_5	p
0	1	0	1	1	−3
1	1	2	2	2	−8

Table 4.6: *Charges for the sheaf*

with one fermionic gauge symmetry. Again the bundle is singular over the set S consisting of four points leading again to a reflexive sheaf. The generations and antigenerations turn out to be identical to model A, providing some evidence that model A and B are in fact dual to each other.

5. Conclusion and Outlook

In this paper, by studying some generic examples it has been argued that the existence of a Landau-Ginzburg phase is not essentiell for $(0, 2)$ target space duality. Instead it suffices that a potentially dual pair shares some non-geometric locus in the extended Kähler moduli space. For a couple of such dual pairs it has been shown that even in the geometric large radius phase the dimensions of the total moduli spaces agree. Furthermore, a brief survey of the adequate mathematical formalism for dealing with more general coherent sheaves was presented. Using these methods to compute part of the massless spectrum some evidence was provided, that relaxing vector bundles to reflexive sheaves does not

change the duality picture at all. The painstaking task of computing the exact number of sheaf deformations has to await a more ambitious attempt.

It would also be interesting to investigate what the F-theory dual picture is for this heterotic duality. Apparently, the elliptic fibers of the heterotic threefolds map under the duality transformation as

$$\mathbb{P}_{1,2,3}[6] \to \mathbb{P}_{1,1,2}[4] \to \mathbb{P}_{1,1,1}[3] \to \mathbb{P}_{1,1,1,1}[2,2]. \tag{5.1}$$

Momentarily, it is not even known what the F-dual fourfold for the last three elliptic fibers in (5.1) are.

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